height of the vessel; T_0, initial temperature of the medium; ρ_0, density at $T = T_0$; P, pressure; p, Boussinesq quasipressure; $\chi = k/\rho_0 c_p$, thermal-diffusivity coefficient; k, thermal-conductivity coefficient: ν, kinematic viscosity coefficient; c_p, specific heat at constant pressure; β, heat expansion coefficient; g, gravitational acceleration; $\vec{V} = (V_x, V_y, 0)$, velocity vector; R, universal gas constant; μ, molecular weight; a, laser beam radius; ψ, stream function; ω, vortex; $q_{Re} = (\alpha I_0 a^3 \beta g)/(\rho_0 c_p \nu^3)$, $T_{Re} = [\beta g(T - T_0)a^3]/\nu^2$, $V_{Re} = Va/\nu$, dimensionless quantities with a as the length scale; Pr, Prandtl number; Δy_V, Δy_T, displacement of maxima of velocity or temperature from beam axis; V^*, T^*, ψ^*, maximal values of velocity, temperature and stream function; r, φ, polar coordinates; V°, T°, values on the ray axis, n, medium refraction index.

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NONLINEAR STABILITY OF MOTION OF VISCOUS LIQUID BETWEEN CONCENTRIC ROTATING CYLINDERS

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The nonlinear stage of the growth of perturbations in the hypercritical region is investigated in the case of a viscous liquid motion in the gap between two rotating cylinders by using the balance method for perturbation energy.

The stability problem of motion of a viscous liquid in the gap between two rotating cylinders has a special place in the theory of hydrodynamic stability. First, the instability of the rotatory Couette flow is one of the two original types of hydrodynamic instability presented by a simple kind of motion. Second, there are available extensive and sufficiently reliable experimental data for this problem; this is especially important when solving a nonlinear problem since in this case the only test of the authenticity of the theoretical conclusions is their agreement with the experimental results.

The linear stability theory of liquid motion for the system under consideration is well known [1]. We shall not dwell on surveying the literature on this subject but shall only mention that in [2, 3] it was rigorously demonstrated that for suitably high Reynolds numbers the Couette circular motion is unstable. In [4, 5] it was shown that in the linear theory, which is limiting in the sense of the ratio of the radii and the ratio of the cylinder

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angular velocity approaching unity, the stability problem of convection arises in the horizontal liquid layer heated from below. In [6] the validity of this assertion was proved also for the nonlinear case and a rigorous motivation was provided for using the Lyapunov—Schmidt method in the analysis of Taylor vertices.

In the present article nonlinear growth of disturbances in the hypercritical region is analyzed. The interaction between the disturbances and the main flow is taken into account by using the energy-balance method for the disturbances described by Stewart in [7]; the basis for this method is the idea originally ascribed to Landau [8] to construct a differential equation involving time for the amplitudes of finite disturbances whose solution approaches asymptotically a constant limit in the hypercritical region. The finite disturbance, as in the Lyapunov—Schmidt method, is given in the form of a product of time-dependent amplitude and a coordinate function which is of the same form as the first eigenvector for the corresponding linear problem.

In [7] the differential equation for the amplitudes of finite disturbances as well as actual evaluations of the torque are carried out for the case of a small gap between the cylinders. In the present article this approach is extended to the case of arbitrary gap size between the cylinders. An attempt has also been made to include finite disturbances within the framework of the method which as regards coordinate dependence agree with the second or third eigenvector of the linear problem and remain within the range of suitably high Reynolds numbers.

1. Balance Equation for Disturbance Energy

The flow of a viscous, incompressible fluid is now considered in the gap between two vertical cylinders with radii and rotation angular velocities given by R_1, Ω_1 and R_2, Ω_2 for the inner and outer cylinder, respectively. It is assumed on the basis of the known experimental results that the arising small perturbations are periodic along the vertical z axis and that they possess circular symmetry (all the quantities are independent of the angular coordinate q).

The velocity V of the motion of fluid and the pressure p are represented as sums,

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}'; \; p = p_0 + p'; \qquad (1)$$

$$\mathbf{v}_0 = \{0, \bar{v}_0(r, t), 0\}; \; \mathbf{v}' = \{u', v', w'\},$$

where V_0, p_0 are the velocity and pressure of the original motion averaged over the z axis, the motion being modified by finite fluctuating perturbations, and V' is the velocity of the finite ampiitude disturbance whose components u', v', w' are functions of the coordinates r, z and of time t. A similar relation is also adopted for the pressure perturbation p'. Following [7] one assumes that the average over z (bar on top) of the perturbations V', p' vanishes ($\bar{V}' = \bar{p}' = 0$).

The equations of motion for an incompressible and viscous fluid and for dimensionless quantities are now written as

$$\frac{1}{\mathrm{Re}} \cdot \frac{\partial \mathbf{v}}{\partial t} + \operatorname{rot} \mathbf{v} \times \mathbf{v} + \frac{1}{2} \operatorname{grad} \mathbf{v}^2 = -\operatorname{grad} p - \frac{1}{\mathrm{Re}} \operatorname{rot} \operatorname{rot} \mathbf{v}, \qquad (2)$$

$$\operatorname{div} \mathbf{v} = 0.$$

One adopts the values $d = R_2 - R_1$, $R_1\Omega_1$, d^2/ν, $\rho R_1 \Omega_1$ as characteristic units for length, velocity, time, and pressure, respectively. The first equation in (2) has a suitable form for projecting it in any direction and, in particular, on the orthogonal axes of the cylindrical coordinate system (r, φ, z) with suitable velocity components (u, v, w).

If into (2) one now substitutes (1) and one averages over z, then (since $\bar{V}' = 0$) the following equation is obtained for the averaged motion:

$$\frac{1}{\mathrm{Re}} \frac{\partial \mathbf{v}_0}{\partial t} + \operatorname{rot} \mathbf{v}_0 \times \mathbf{v}_0 + \overline{\operatorname{rot} \mathbf{v}' \times \mathbf{v}'} + \frac{1}{2} \operatorname{grad} (\mathbf{v}_0^2 + \mathbf{v}'^2) = \operatorname{grad} p_0 - \frac{1}{\mathrm{Re}} \operatorname{rot} \operatorname{rot} \mathbf{v}_0. \qquad (3)$$

The continuity equation for V_0 is satisfied identically.

The boundary conditions for (3) consist of the tangential component of the velocity $\bar{v}_0$ of the averaged motion on the boundary of the cylinders equal to the corresponding linear velocities of the points of cylinder surfaces, namely,

$$\bar{v}_0(r, t) = 1 \text{ for } r = r_1,$$
$$\bar{v}_0(r, t) = m\beta \text{ for } r = r_2. \tag{4}$$

By inserting (1) into (2) and taking into account Eq. (3) one obtains a system of equations of motion for the perturbations

$$\frac{1}{\mathrm{Re}}\cdot\frac{\partial \mathbf{v}'}{\partial t} + \mathrm{rot}\,\mathbf{v}' \times \mathbf{v}_0 + \mathrm{rot}\,\mathbf{v}_0 \times \mathbf{v}' + \mathrm{rot}\,\mathbf{v}' \times \mathbf{v}' - \overline{\mathrm{rot}\,\mathbf{v}' \times \mathbf{v}'} +$$
$$+ \frac{1}{2}\,\mathrm{grad}\,(\mathbf{v}'^2 - \overline{\mathbf{v}'^2}) + \mathrm{grad}\,(\mathbf{v}_0\mathbf{v}' - \overline{\mathbf{v}_0\mathbf{v}'}) = -\mathrm{grad}\,p' - \frac{1}{\mathrm{Re}}\,\mathrm{rot}\,\mathrm{rot}\,\mathbf{v}', \tag{5}$$
$$\mathrm{div}\,\mathbf{v}' = 0$$

with vanishing boundary values for V' on the cylinder surfaces.

The balance equation for the perturbation energy can be found, for example, in [7]. However, it can be derived directly from Eq. (5) if one takes a scalar product of the latter with V' and then integrates it over the entire volume. By taking into account the axial symmetry as well as the periodicity in z of the perturbations the integration over the volume reduces to the integration over the area of an axial section of the gap with the height along z equal to the wavelength λ for a given perturbation. It can easily be seen that the third and the fourth components on the left are lost since they are scalar products of orthogonal vectors. All the components containing gradients also vanish in view of the vanishing boundary values of the velocity normal component since the volume integrals of these expressions can be transformed into surface integrals with the continuity equation taken into account. The integral of the expression $(\overline{\mathrm{rot}\,V' \times V'})\cdot V'$ also vanishes since the expression in the brackets is independent of z and the integration along z gives the mean value $\bar{V}'$ which vanishes by definition. Transforming the remaining integrals one obtains the following balance equation for the perturbation energy:

$$\frac{\partial}{\partial t}\int_{r_1}^{r_2}\int_0^{\lambda}\frac{\mathbf{v}'^2}{2}\,r\,dr\,dz = \mathrm{Re}\int_{r_1}^{r_2}\int_0^{\lambda}(-\overline{u'v'})\left(\frac{\partial \bar{v}_0}{\partial r} - \frac{\bar{v}_0}{r}\right)r\,dr\,dz - \int_{r_1}^{r_2}\int_0^{\lambda}(\mathrm{rot}\,\mathbf{v}')^2\,r\,dr\,dz; \tag{6}$$
$$(\mathrm{rot}\,\mathbf{v}')_r = -\frac{\partial \mathbf{v}'}{\partial z};\quad (\mathrm{rot}\,\mathbf{v}')_\varphi = \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial r};\quad (\mathrm{rot}\,\mathbf{v}')_z = \frac{1}{r}\,\frac{\partial (rv')}{\partial r};$$
$$\mathbf{v}'^2 = u'^2 + v'^2 + w'^2.$$

The term on the left of Eq. (6) represents the increase of the perturbation energy in the volume under consideration. The first term on the right is the integral of the product of the Reynolds stress by the velocity gradient of the averaged motion; it represents the energy content transmitted from the averaged motion to the perturbations in a unit of time. The second integral on the right represents the energy loss of the perturbations per unit of time due to dissipation. This term is always positive. If, due to the Reynolds stresses, the averaged motion changes in such a way that the energy losses of the perturbations are exactly compensated by the energy passing from the averaged motion to the perturbations then a steady secondary flow exists in these conditions.

2. Linearized Problem

For infinitely small perturbations the second-order terms in V' in (5) can be ignored and linear equations of motion are obtained in the form

$$\frac{1}{\mathrm{Re}}\cdot\frac{\partial \mathbf{v}'}{\partial t} + \mathrm{rot}\,\mathbf{v}' \times \mathbf{v}_0 + \mathrm{rot}\,\mathbf{v}_0 \times \mathbf{v}' = -\mathrm{grad}\,p' - \frac{1}{\mathrm{Re}}\,\mathrm{rot}\,\mathrm{rot}\,\mathbf{v}', \tag{7}$$
$$\mathrm{div}\,\mathbf{v}' = 0.$$

If one writes down the projections of Eq. (7) on the cylindrical coordinate axes, introduces the flow function in the (r, z) plane according to the relations

$$u' = -\frac{1}{r}\cdot\frac{\partial (r\psi')}{\partial z},\quad w' = \frac{1}{r}\,\frac{\partial (r\psi')}{\partial r} \tag{8}$$

and eliminates the pressure, then one obtains for the functions ψ', v' the following system of equations:

$$\left(\nabla^2 - \frac{1}{r^2}\right)\frac{\partial\psi'}{\partial t} + 2\,\mathrm{Re}\,\frac{v_0}{r}\cdot\frac{\partial v'}{\partial z} = \left(\nabla^2 - \frac{1}{r^2}\right)^2\psi',$$
$$\frac{\partial v'}{\partial t} - \left(\frac{\partial v_0}{\partial r} - \frac{v_0}{r}\right)\frac{\partial\psi'}{\partial z} = \left(\nabla^2 - \frac{1}{r^2}\right)v'. \tag{9}$$

One analyzes now the solutions periodic in z and exponential in time,

$$\psi' = -\exp(\sigma t)\sin(kz)\,\psi(r),\quad v' = \exp(\sigma t)\cos(kz)\,v(r), \tag{10}$$

and using the expression for the velocity of the unperturbed motion,

$$v_0 = A_0 r + \frac{B_0}{r};\ A_0 = \frac{1-\beta m^2}{r_1(1-m^2)};\ B_0 = \frac{r_1(1-\beta m^2)}{m^2-1}, \tag{11}$$

one obtains from (9) an eigenvalue problem for the eigenvalues σ:

$$(L-k^2)\psi - \varphi = 0,$$
$$(L-k^2)\varphi - \sigma\varphi = 2k\,\mathrm{Re}\,\omega(r)\,v, \tag{12}$$
$$(L-k^2)v - \sigma v = 2kA_0\psi,\ \omega(r) = A_0 + \frac{B_0}{r}.$$

with the boundary conditions for the functions ψ, v given by

$$v = \psi = \frac{\partial\psi}{\partial r} = 0 \text{ for } r = r_1,\ r_2.$$

To solve the latter problem the numerical method as described in [10] is used to find the first three eigenvalues σ_i ($i = 1, 2, 3$) and the corresponding eigenvectors (ψ_i, φ_i, v_i).

3. Torque

The averaged equation (3) of motion projected on the cylindrical coordinate axes can be written as

$$\frac{1}{r}\cdot\frac{\partial}{\partial r}(r\overline{u'^2}) - \frac{\bar{v}_0^2 + \overline{v'^2}}{r} = -\frac{\partial p_0}{\partial r},$$
$$\frac{\partial\bar{v}_0}{\partial t} + \frac{\mathrm{Re}}{r^2}\cdot\frac{\partial}{\partial r}(r^2\overline{u'v'}) = \frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial(r\bar{v}_0)}{\partial r}\right), \tag{13}$$
$$\frac{1}{r}\cdot\frac{\partial}{\partial r}(r\overline{u'w'}) = -\frac{\partial p_0}{\partial z}.$$

The first equation of (13) determines the radial pressure gradient necessary to equalize the centrifugal force (the second term on the left) and the Reynolds stress. In a similar manner the third equation determines the axial pressure gradient which in our case vanishes since within the framework of the adopted assumptions as regards the coordinate dependence of the finite perturbations the averaged-over-z value $\overline{u'w'} = 0$, as will be seen from our further considerations.

The second equation of (13) gives the velocity component $\bar{v}_0$ of the averaged motion against the Reynolds stress. The time dependence of $\bar{v}_0$ must be maintained in the general case since with the perturbation energy growing or fading the averaged motion also varies in accordance with the law of the energy conservation (6). For steady averaged motion ($\partial\bar{v}_0/\partial t = 0$) provided the quantity $\overline{u'v'}$ is a known function of the coordinates one can write the solution of the second equation in (13) as follows:

$$\bar{v}_0 = Ar + \frac{B}{r} + \mathrm{Re}\,r\int_{r_1}^{r}\frac{\overline{u'v'}}{r}\,dr,$$
$$A = A_0 - b\,\mathrm{Re}\int_{r_1}^{r_2}\frac{\overline{u'v'}}{r}\,dr;\ B = B_0 + b\,\mathrm{Re}\int_{r_1}^{r_2}\frac{\overline{u'v'}}{r}\,dr;\ b = \frac{m^2 r_1^2}{m^2-1}, \tag{14}$$

where A_0, B_0 are given by the relations (11).

The relevant experimental results which were analyzed in [9] are consistent with the concept of dynamic equilibrium in the hypercritical domain of averaged motion and finite-amplitude perturbations. The approximate method for evaluating this equilibrium state consists of the following:

It is assumed that the time variation of the velocity of the averaged motion $\bar{v}_0$ in Eq. (13) can be ignored; for low eigenvalues σ_1 and for short times the latter follows directly from the linear stability theory. It is considered that this condition also holds for longer times, which is confirmed by the experiments on steady equilibrium motions in the hypercritical domain. In this case the tangential stress function in the balance equation (6) can be determined from the relation (14).

It is also assumed that the spatial distribution of the perturbation velocity is given in accordance with the linear theory; then the velocity components are expressed by the relations (8) and (10) in which the exponent is replaced by a finite amplitude $a_1(t)$ which is time dependent. Substituting these relations into Eqs. (6) for the energy balance and transforming the integrals thus derived by using the linear equations (12) one obtains for the square $a_1^2(t)$ of the finite amplitude of disturbance the equation

$$\frac{da_1^2(t)}{dt} = 2\sigma a_1^2(t) + \frac{2B_0\alpha_1}{\gamma_1} k_1 (\mathrm{Re} - \mathrm{Re}_1)\, a_1^2(t) - \frac{\beta_1 - 2b\alpha_1^2}{2\gamma_1} k_1^2 \mathrm{Re}_1^2 a_1^4(t). \tag{15}$$

In the above Re_1 is the Reynolds number for which one evaluates the eigenvalue σ_1 and the corresponding eigenvector-function (ψ_1, φ_1, v_1) of the problem (12) whose components are used in evaluating the integrals

$$\gamma_1 = \frac{1}{2}\int_{r_1}^{r_2} (v_1^2 - \varphi_1\psi_1)\, r\,dr;\quad \alpha_1 = \int_{r_1}^{r_2} \frac{v_1\psi_1}{r}\, dr;\quad \beta_1 = \int_{r_1}^{r_2} v_1^2\psi_1^2 r\,dr. \tag{16}$$

The nonlinear equation (15) possesses a solution which for $t \to \infty$ approaches the time-independent limit given by

$$a_1^2 = \frac{4\sigma_1\gamma_1}{k_1^2\mathrm{Re}_1^2(\beta_1 - 2b\alpha_1^2)} + \frac{4B_0\alpha_1(\mathrm{Re} - \mathrm{Re}_1)}{k_1\mathrm{Re}_1^2(\beta_1 - 2b\alpha_1^2)}. \tag{17}$$

If Re_1 is the Reynolds number for neutral perturbations with a given wave number k_1 (in particular, these numbers can be equal to their critical values, k_1^*, Re_1^*), then in this case one has $\sigma_1 = 0$ and the square of the equilibrium amplitude a_1^2 is determined only by the second term of (17) depending linearly on the difference $\mathrm{Re} - \mathrm{Re}_1$. However, if the coefficients (16) are determined by means of the components of the first eigenvector for a given k_1 and for the current value of Re, then the second term in (17) vanishes in view of $\mathrm{Re}_1 = \mathrm{Re}$ and the square of the equilibrium amplitude can be determined only by means of the first term proportional to the eigenvalue σ_1. In the latter case for nonlinearity to be taken into account the approach is more indirect since it is possible to consider all the feasible modifications of integrals in (16) for various values of Re. However, the computations have shown that the results of both variants differ only insignificantly. It is more convenient in practice to use in the computations only the second term for the square (17) of the equilibrium amplitude, having determined the coefficients in (16) only once, for example, for the critical values k_1^*, Re_1^* of the parameters.

The mean torque transmitted to the rotating fluid by means of friction on the outer cylinder of given length h is determined by the relation

$$G = 2\pi R_1^2 h\mu \left(\frac{d\bar{v}_0}{dr} - \frac{v_0}{r} \right)_{r=r_1} \frac{\Omega_1 R_1}{d}. \tag{18}$$

The tangential stress is now determined from the formula (14) and the result is substituted into (18); this yields

$$G_1 = C\,\mathrm{Re}^2 \left(\frac{2B_0}{\mathrm{Re}} + bk_1\alpha_1 a_1^2 \right);\quad C = 2\pi h\rho\nu^2. \tag{19}$$

The first term of the right-hand side in the formula (19) characterizes the torque of the tangential stresses of the laminar flow by ignoring the perturbations; the second term

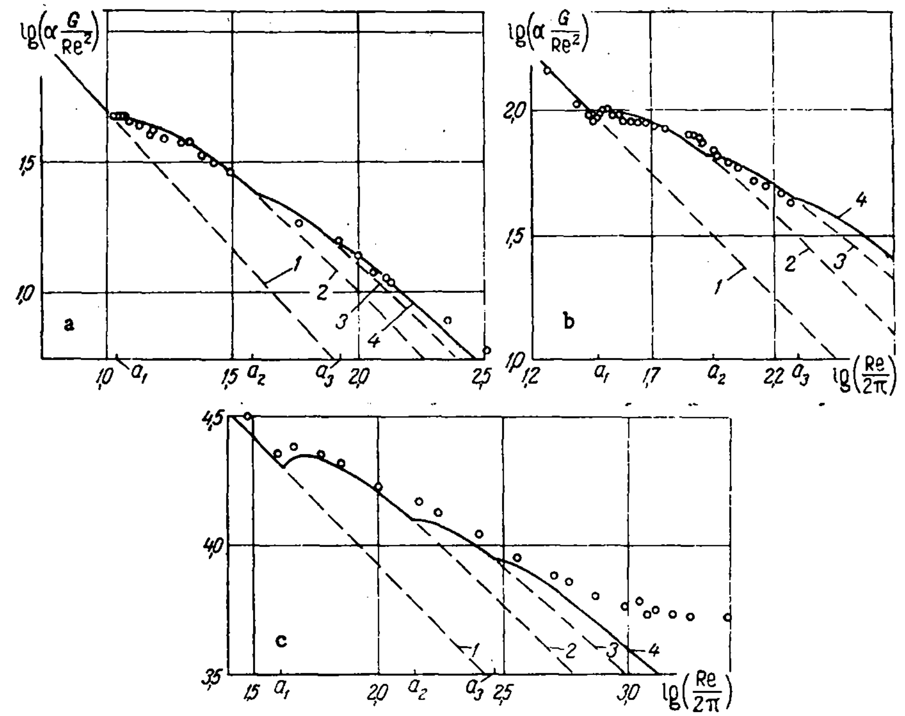

Fig. 1. Torque G in $g{\cdot}cm^2/sec^2$ against Reynolds number: 1) theoretical curve of unperturbed laminar motion; 2, 3, 4) computed curves obtained by successively taking into account the effect of one, two, or three perturbation modes on the averaged motion with the Reynolds number passing through the value of the first, the second, and the third critical number, respectively, designated in the diagram as the coordinates a_1, a_2, a_3. For a: $R_1 = 1$ cm; $R_2 = 2$ cm; h = 5 cm; $\rho = 0.8404$ g/cm^3; $\gamma = 0.1226$ cm^2/sec; $k_1^* = 3.16$; $k_2^* = 5.38$; $k_3^* = 7.0$; $\alpha = 4\pi^2 d^2 R_1^2/\rho\gamma = 3125.5$ cm^5; experimental points according to Donnelly ([9], Table 2). For b: $R_1 = 1.9$ cm; $R_2 = 2$ cm; h = 5 cm; $\nu = 5.796 \cdot 10^{-3}$ cm^2/sec; $\rho = 1.585$ g/cm^3; $k_1^* = 3.13$; $k_2^* = 5.3$; $k_3^* = 7.0$; $\alpha = 26{,}766$ cm^5; experimental points according to Donnelly ([9], Table 1). For c: $R_1 = 3.94$ cm; $R_2 = 4.05$ cm; h = 84.4 cm; $\nu = 0.131$ cm^2/sec; $\rho = 1.171$ g/cm^3; $k_2^* = 3.14$; $k_1^* = 5.3$; $k_3^* = 7.0$; $\alpha = 369.01$ cm^5; experimental points according to Taylor ([9], Fig. 2).

represents the averaged supplement to the torque which is due to steady perturbations of finite amplitude.

Numerical computations for the steady outer cylinder using the formula (19) were carried out for three cases in conformity with the experimental measurements given in [9]. The results are shown in the diagram (curves 2). One can see that there is a good agreement between the solution and the experimental points in the first hypercritical domain ($Re_1^* \leqslant Re \leqslant Re_2^*$) for all three cases with various ratios of the cylinder radii. It is recalled that by Re_i^* (i = 1, 2, 3, ...) one understands a number of critical values for the Reynolds numbers determined from the first, the second, the third, etc., eigenvectors, respectively, of the corresponding linear problem (12). It can also be seen from the diagram that the deviation of the theoretical curves from the experimental points occurs exactly at the passing of the Reynolds number through the second critical value Re_2^* and grows with the increasing Reynolds number. It can be assumed therefore that for $Re > Re_2^*$ (in this domain already the second moment $\sigma_2 > 0$, in accordance with the linear theory) the equilibrium state is established for which the energy transmitted from the averaged motion to the perturbations is divided between two disturbance modes corresponding coordinatewise to the first or the second eigenvector of the linear problem (12). An approximate evaluation of the equilibrium amplitudes for the two disturbance modes can be accomplished by using the following procedure.

A perturbation in the Reynolds numbers domain $\mathrm{Re}_2^* \le \mathrm{Re} \le \mathrm{Re}_3^*$ is now written as a sum of two perturbations, $\mathbf{v}' = \mathbf{v}_1' + \mathbf{v}_2'$ (this also applies if only one perturbation mode is considered), and it is assumed that the coordinate form of two perturbation modes $\mathbf{v}_1'$, $\mathbf{v}_2'$ is determined by the first eigenvector $\mathbf{v}_1$ and the second eigenvector $\mathbf{v}_2$, respectively, for the linear problem (12), namely,

$$\mathbf{v}' = a_1(t)\,\mathbf{v}_1 + a_2(t)\,\mathbf{v}_2,$$

where the amplitudes of these two modes of finite perturbations a_1 and a_2 are to be determined.

It is now assumed that the type of growth of each of the two perturbation modes is specified by its interaction with the average motion relative only to a specified perturbation mode irrespective of the existence of the other mode. Under this assumption the balance equation for the perturbation energy can be applied for each mode separately. One then obtains Eq. (15) for the square of the amplitude of the first mode and a similar equation for the square of the amplitude $a_2^2(t)$ for the second mode; in the latter case one has to replace the subscript 1 everywhere by 2, which indicates that one employs the components of the second eigenvector of the problem (12) to determine the former by using the formulas (16).

In this case the formula (18) for the torque is transformed into

$$G_2 = C\,\mathrm{Re}^2\left[\frac{2B_0}{\mathrm{Re}} + b\,(k_1\alpha_1 a_1^2 + k_2\alpha_2 a_2^2)\right] \tag{20}$$

if the velocity of the averaged motion is obtained from the relation (14) and one takes into account the two perturbation modes.

In a similar manner the additional contribution to the mean torque is taken into account in the third perturbation mode; the latter may exist in the third hypercritical domain for $\mathrm{Re} > \mathrm{Re}_3^*$ [where the third eigenvalue of the problem (12) is also positive, $\sigma_3 > 0$]. The coordinate relation for this perturbation mode is assumed to be equal to the third eigenvector of the problem (12). The formula for the mean torque differs in this case from (20) by an additional term $k_3\alpha_3 a_3^2$ in the parentheses on the right.

In evaluating the torque (20), the squares a_i^2 (i = 1, 2, 3) of the equilibrium amplitudes were determined using the second term of the formula (17) for the coefficients α_i, β_i (i = 1, 2, 3) which, in turn, were determined by using the components of the corresponding eigenvectors of the linear problem (12) for the critical values k_i^*, Re_i^* (i = 1, 2, 3).

The results of the computations are shown in Fig. 1. Curve 1 is the linear characteristic of the laminar motion, 2, 3, and 4 are the computed curves obtained by taking successively into account the interactions with the averaged motion of one, two, or three perturbation modes, respectively.

NOTATION

R_1, R_2 and Ω_1, Ω_2, radii and angular velocities of inner and outer rotating cylinders, respectively; $d = R_2 - R_1$, the gap width between cylinders; $r_i = R_i/d$ (i = 1, 2), dimensionless cylinder radii; ρ, ν, μ, density, coefficients of kinematic and dynamic viscosity, respecitively; h, cylinder length; $\mathrm{Re} = dR_1\Omega_1\ \nu$, Reynolds number; Re_i^* (i = 1, 2, 3), critical Reynolds numbers; G, torque; k_i, wave numbers; σ_i, eigenvalues of linear problem (i = 1, 2, 3); $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$; $L = \frac{d^2}{dr^2} + \frac{1}{r}\cdot\frac{d}{dr} - \frac{1}{r^2}$; $m = R_2/R_1$; $\beta = \Omega_2/\Omega_1$.

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METHOD OF FINITE ELEMENTS FOR SOLVING SOME HEAT-CONDUCTION PROBLEMS

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Heat-conduction problems are investigated with the aid of a new variational method, namely, the method of finite elements (MFE).

Heat processes in constructions or in mechanical equipment with high-temperature gradients are at present studied in a number of investigations. For a theoretical solution of the problems thus arising one can employ, in principle, either analytic or numerical methods. The classical analytic methods, including the methods based on integral transformations, can produce satisfactory solutions for simple physical models; their use, however, in involved problems, in practice, is rather doubtful. The numerical methods employed until recently were almost exclusively based on the method of finite differences. Variational methods have at present found wide application (in particular, the MFE), since the use of the latter results in matrix equations suitable for processing on digital computers.

The main concept of the MFE consists in subdividing the entire solution domain into a set of a finite number of elements, the links between adjacent elements being provided in a finite number only of the so-called points of contact. The continuous solution of the original problem in the old domain (for heat conduction the latter is the temperature field) is replaced by a piecewise polynomial one with values specified in advance at the nodes of the complex. Since these values are the same for adjacent elements therefore continuity of the solution is attained in the entire domain under investigation. Some of the main advantages of the MFE are the ease of satisfying any boundary conditions for bodies of quite different shapes including holes and complicated boundaries and also that any inhomogeneities or anisotropy can be taken into account, and finally that one can solve nonlinear problems with the aid of various iteration procedures.

The unsteady heat-conduction equation can be written as follows:

$$\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left[\lambda_{ij}(x,\ t,\ \theta)\frac{\partial\theta}{\partial x_j} - \rho(x,\ t,\ \theta)\,c(x,\ t,\ \theta)\,\dot{\theta} + Q(x,\ t,\ \theta) = 0,\right. \tag{1}$$

where $\dot{\theta} = \partial\theta/\partial t$ with the boundary and initial conditions

$$\theta(x,\ 0) = \theta^0(x),\ \frac{\partial\theta}{\partial n} + \alpha(x,\ t,\ \theta)(\theta - \theta_m)|_{A^\alpha} = 0,$$
$$\theta|_{A^\theta} = \theta^A(x,\ t),\quad \frac{\partial\theta}{\partial n} + q_p(x,\ t,\ \theta)|_{A^q} = 0, \tag{2}$$

where $A = A^\theta \cup A^q \cup A^\alpha$ is the boundary of the domain Ω under investigation; $\partial\theta/\partial n$ is the normal derivative to the surface and α is the heat-emission coefficient. Of the variational methods employed in heat problems the Galerkin method is the one most often used [1]. One considers the basic space of the functions φ such that

$$\varphi \in H \text{ and } \varphi|_{A^\theta} = 0,$$

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